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A framework for the calculation of the $\Delta N\gamma^*$ transition form factors on the lattice

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Abstract

Using the non-relativistic effective field theory framework in a finite volume, we discuss the extraction of the $\Delta N\gamma^*$ transition form factors from lattice data. A counterpart of the Lüscher approach for the *matrix elements* of unstable states is formulated. In particular, we thoroughly discuss various kinematic settings, which are used in the calculation of the above matrix element on the lattice. The emerging Lüscher–Lellouch factor and the analytic continuation of the matrix elements into the complex plane are also considered in detail. A full group-theoretical analysis of the problem is made, including the partial-wave mixing and projecting out the invariant form factors from data.

1. Introduction

In recent years, the calculation of the $\Delta N\gamma^*$ transition form factors on the lattice has been carried out, see Refs. [1–3]. The electromagnetic, axial and pseudoscalar form factors of the

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Δ-resonance have been also studied [4,5]. It should be noted, however, that in these simulations the quark mass values are large enough so that the Δ is a stable particle and thus using the standard formalism for the analysis of the lattice data on these form factors is justified. On the other hand, lattice simulations with physical quark masses have already been performed. At such quark masses, the Δ is not stable anymore and the data should be analyzed properly to extract the parameters of the resonance (see, e.g., [6]).

It is well known that resonances cannot be identified with isolated energy levels in lattice QCD simulations which are necessarily performed in a finite volume. In order to determine the mass and width from the measured spectrum, one first extracts the scattering phase shift by using the Lüscher equation [7]. At the next step, using some parameterization for the K-matrix (e.g., the effective-range expansion), a continuation into the complex energy plane is performed. Resonances correspond to the poles of the scattering T-matrix on the second Riemann sheet, and the real and imaginary parts of the pole position define the mass and the width of a resonance. This is a pretty standard procedure that has been used in a number of recent papers [8–10]. A generalization of the approach to moving frames has been first proposed in Ref. [11], and a full group-theoretical analysis of the Lüscher equation in moving frames, including the issues related to the scattering of the spin-non-zero particles, has been carried out, e.g., in Refs. [12–18]. An alternative, albeit a closely related procedure consists in fitting the data to the energy spectrum by using unitarized ChPT in a finite volume [19–21]. The above approach qualitatively amounts to a parameterization of the K-matrix through the solution of the equations of the unitarized ChPT (in the infinite volume) that may *a priori* have a larger range of applicability than the effective-range expansion. Note also that the approach has been generalized to the multichannel scattering case [22–28]. An analysis of the two-channel case has been carried already out for the toy model in $1+1$ dimensions [29] and should now be applied in physically interesting cases. Further, using twisted boundary conditions [30–32] to facilitate the accurate extraction of the resonance parameters has been advocated, e.g., in Refs. [19,20, 23], and the possibility of the partial twisting has been investigated in Refs. [31,33,34]. Last but not least, recently an extension of the Lüscher approach to the 3-particle case has been proposed by several groups [35–38], albeit there is still much more work required in this direction.

As one sees from the above discussion, up to date the framework for the extraction of the resonance parameters (the mass and the width) from lattice data is well established (at least, for the resonances that do not decay into three or more particles). For the calculation of the more complicated quantities, e.g., the resonance form factors, the old approach that treats the resonance as a stable state, is still widely used, albeit it is clear that the same problems arise also here. What one needs is a *generalization of the Lüscher finite-volume approach to the form factors.* In our recent papers [39,40], we have formulated such a generalization, considering the form factor of a spinless resonance. The procedure of extracting the resonance form factor from the *form factors of the eigenstates of the Hamiltonian*, which are actually measured on the lattice, closely resembles the procedure of extracting the resonance pole position and also implies the analytic continuation into the complex energy plane by using, e.g., the effective range expansion. There are, however, differences as well. Most notably, as was shown in Ref. [40], in $3+1$ dimensions, due to the presence of the so-called finite fixed points, the procedure of the analytic continuation and taking the infinite-volume limit is no more straightforward and measuring the form factor for at least two different energy levels is required, in order to achieve an unambiguous extraction of the resonance form factor.

The present paper is a continuation and the generalization of Refs. [39,40] in two aspects:

(i) In Refs. [39,40], an elastic resonance form factor (an example: the electromagnetic form factor $\Delta\Delta\gamma^*$) has been considered. In this paper, we address transition form factors, in particular, $\Delta N\gamma^*$ that is more interesting from the phenomenological point of view. It turns out that the presence of one stable particle in the out-state (here, the nucleon) leads to crucial simplifications. As we shall demonstrate, the finite fixed points do not exist in this case, so the measurement of a single energy level (albeit at several volumes) will suffice. We shall discuss in detail various lattice settings, which provide an access to the measurement of the form factor.

(ii) All particles and currents, considered in Refs. [39,40], were scalar. On the other hand, the particles, whose form factors we want to calculate, have spin. In this paper, we consider the inclusion of spin into the formalism and carry out a full group-theoretical analysis of the obtained equations along the lines described in Ref. [16].

The paper is organized as follows. In Section 2, we start from defining the resonance form factor in the infinite volume and discuss the analytic continuation into the complex plane. The projection of various scalar form factors will be considered. In Section 3 we consider the kinematics, which should be used for measuring the matrix elements of a current between the eigenstates of the Hamiltonian. This issue is very important for performing the analytic continuation of the above matrix element, *keeping the relative three-momentum of the photon and nucleon fixed*. Further, in Section 4, we calculate these matrix elements within the non-relativistic effective field theory (EFT) and demonstrate that the finite fixed points are absent, when one of the external particles is stable. The initial-state interactions, which manifest themselves in the Lüscher–Lellouch factor, should be properly included in order to take into account the difference in normalization of the matrix elements in the infinite and in a finite volume. In Section 5 we collect all bits and pieces and formulate a prescription for the extraction of the resonance transition form factor from data. Section 6 contains our conclusions. Finally, in Appendix A, the formulae for the partial-wave expansion of the photoproduction amplitudes are displayed.

2. Resonance form factor in the infinite volume

To the best of our knowledge, the procedure for calculating the matrix elements of operators between the bound state vectors in field theory has been first addressed in Ref. [41] (a very detailed and transparent discussion of the problem can be found in Ref. [42]). In short, the procedure boils down to the following. For simplicity, consider the scalar case first. Let $|P\rangle$ be a stable bound state moving with a four-momentum P_μ with $P_\mu P^\mu = M_B^2$. The fact that this is a bound state and not an elementary state is equivalent to the statement that $\langle 0|\phi(x)|P\rangle = 0$, where $\phi(x)$ stands for any field which is present in the Lagrangian. Consider now an operator $O(X)$ built from the elementary fields ϕ. The operator $O(X)$ can be either local or non-local. In the latter case, X denotes the center-of-mass coordinate of the fields entering in $O(X)$. The only requirement on this operator is that $\langle 0|O(X)|P\rangle \neq 0$. The Fourier-transform of the two-point function of the operators O has a pole at $P^2 = M_B^2$:

$$i\int d^4X e^{iPX}\langle 0|T\,O(X)\bar{O}(0)|0\rangle = \frac{Z_B}{M_B^2 - P^2} + \text{regular terms at } P^2 \to M_B^2, \tag{2.1}$$

where Z_B is the wave function renormalization constant of the bound state (for the scalar operators, considered here, the conjugated operator $\bar{O} = O^\dagger$).

Let us now consider the three-point function with any local operator J (the "current"). This function has a double pole

$$F(P,Q) = i^2 \int d^4X d^4Y e^{iPX-iQY} \langle 0|T\,O(X)J(0)\bar{O}(Y)|0\rangle$$
$$= \frac{Z_B^{1/2}}{M_B^2 - P^2} \langle P|J(0)|Q\rangle \frac{Z_B^{1/2}}{M_B^2 - Q^2} + \cdots, \tag{2.2}$$

where the ellipses stand for the less singular terms. From this equation one immediately sees that the matrix element of a current J between the bound state vectors is defined through

$$\langle P|J(0)|Q\rangle = \lim_{P^2,Q^2 \to M_B^2} Z_B^{-1}\big(M_B^2 - P^2\big)\big(M_B^2 - Q^2\big)F(P,Q). \tag{2.3}$$

Due to the Lorenz-invariance, the matrix element in the l.h.s. of this equation is a function of a single scalar variable $t = (P-Q)^2$.

Note also that the expression given in Eq. (2.3) defines the matrix element between bound states. In a completely similar manner, it is possible to define the matrix elements between a bound state and an elementary state, or between two different bound states – all differences boil down to the proper choice of the operators O.

In case of a *resonance* rather than a bound system, no corresponding single-particle state exists in the Fock space. In the literature, one encounters two different approaches to the problem. One approach, which implies the definition of the form factor from the amplitudes measured at *real energies,* invokes the Breit–Wigner parameterization of the resonant amplitude and extracts the resonance formfactors at the energy where the scattering phase shift passes through 90°. Within the second approach, the resonance form factor is defined through the continuation to the resonance pole position in the complex plane, as described below. Let $O(X)$ be the operator with the quantum numbers of a resonance. The two-point function of the operators O develops a pole on the unphysical Riemann sheet in the complex plane

$$i \int d^4X e^{iPX} \langle 0|T\,O(X)\bar{O}(0)|0\rangle = \frac{Z_R}{s_R - P^2} + \text{regular terms at } P^2 \to s_R, \tag{2.4}$$

where the quantities s_R, Z_R are now complex. The real and imaginary parts of $E_R = \sqrt{s_R}$ give the mass and the half-width of the resonance, respectively.

Further, the three-point function develops a double pole in the complex plane, and the *resonance matrix element* of any current J is still defined by a formula similar to Eq. (2.3):

$$\langle P|J(0)|Q\rangle = \lim_{P^2,Q^2 \to s_R} Z_R^{-1}\big(s_R - P^2\big)\big(s_R - Q^2\big)F(P,Q). \tag{2.5}$$

We would like to stress that the quantity on the l.h.s. of Eq. (2.5) is a mere notation for the matrix element: there exists no isolated resonance state $|P\rangle$ in the spectrum. Again, due to the Lorentz-invariance, this quantity is a function of a single variable $t = (P-Q)^2$.

The following questions arise naturally in connection to the procedure described above:

(i) Is it not possible to avoid the analytic continuation into the complex energy plane?
(ii) Experiments can only be performed for real energies. How does one perform the analytic continuation of the experimental data?

In brief, answers to these question are:

(i) Relating the form factor to the measured scattering amplitudes by using, e.g., the Breit–Wigner parameterization, yields a model-dependent result, since the background is not known. Consequently, the form factor, extracted at the real energies, will be process-dependent. This problem does not arise, when an analytic continuation to the resonance pole is performed. The resonance matrix elements extracted through the analytic continuation, are the quantities that characterize the resonance itself and not the process where they were determined.

(ii) The analytic continuation of the experimental data (e.g., in order to extract the magnetic moment of a Δ-resonance) is, in general, a very difficult procedure and is severely limited by the experimental uncertainties. However, the goal is still worth trying, see the arguments above.

It should be mentioned that both definitions of the form factor: on the real axis (see, e.g., [43, 44]) as well as at the resonance pole [45], have been already used for the analysis of the experimental data (the latter work contains also the comparison of the resonance parameters, extracted by using different methods). In order to make it possible to compare lattice calculations with all existing experimental results, in this paper we provide the formulae which should be used on the real axis, as well as in the complex energy plane. Here we stress once more that only the definition, based on the analytic continuation, yields a resonance form factor that is devoid of any process-dependent ambiguities. Further, it will be explicitly demonstrated that both methods yield the same result in the limit of the infinitely small width.

Up to now, all particles and operators considered were scalars. In order to include the resonances of a generic spin, we follow closely the procedure of Ref. [46]. Let $O_\alpha(X)$ be the interpolating field for a resonance. Here, α denotes the collection of indices characterizing a resonance with spin (Dirac indices, vector indices). The two-point function in the vicinity of the resonance pole has the following behavior:

$$i\int d^4Xe^{iPX}\langle 0|T O_\alpha(X)\bar{O}_\beta(0)|0\rangle = \frac{Z_R P_{\alpha\beta}(P,s_R)}{s_R-P^2} + \text{regular terms at } P^2\to s_R, \qquad (2.6)$$

where $P_{\alpha\beta}$ denotes the projector on the positive energy “states”

$$P_{\alpha\beta}=\sum_\varepsilon u_\alpha(P,\varepsilon)\bar{u}_\beta(P,\varepsilon), \qquad (2.7)$$

where the sum runs over spin projections on the third axis (ε) and $u_\alpha(P,\varepsilon)$ denotes the solution of the free wave equation for a particle with a given spin.

Below, we shall give a construction of $u_\alpha(P,\varepsilon)$ in case of the spin-$1/2$ and spin-$3/2$ particles. In case of the spin-$1/2$ particle, this quantity is given by [46]:

$$u(P,1/2)=\sqrt{P^0+E_R}\begin{pmatrix} 1 \\ 0 \\ \frac{P^3}{P^0+E_R} \\ \frac{P^1+iP^2}{P^0+E_R}\end{pmatrix},$$

$$
\begin{aligned}
u(P,-1/2) &= \sqrt{P^0+E_R}\begin{pmatrix} 0 \\ 1 \\ \frac{P^1-iP^2}{P^0+E_R} \\ \frac{-P^3}{P^0+E_R} \end{pmatrix}, \\
\bar{u}(P,1/2) &= \sqrt{P^0+E_R}\left(1, 0, \frac{-P^3}{P^0+E_R}, \frac{-(P^1-iP^2)}{P^0+E_R}\right), \\
\bar{u}(P,-1/2) &= \sqrt{P^0+E_R}\left(0, 1, \frac{-(P^1+iP^2)}{P^0+E_R}, \frac{P^3}{P^0+E_R}\right).
\end{aligned}
\tag{2.8}
$$

These spinors obey the Dirac equations with the complex "mass" $P^2 = s_R$

$$
(\not{P} - E_R)u(P,\varepsilon) = 0, \qquad \bar{u}(P,\varepsilon)(\not{P} - E_R) = 0 \tag{2.9}
$$

as well as the identities

$$
\sum_{\varepsilon} u(P,\varepsilon)\bar{u}(P,\varepsilon) = (\not{P} + E_R), \qquad \bar{u}(P,\varepsilon)u\big(P,\varepsilon'\big) = 2E_R\delta_{\varepsilon\varepsilon'}. \tag{2.10}
$$

Note, however, that, if E_R and P_μ are complex quantities, then, in general,

$$
\bar{u}(P,\varepsilon) \neq u(P,\varepsilon)^\dagger\gamma_0. \tag{2.11}
$$

In case of a particle with a spin-3/2, one has to construct the solutions of the Rarita–Schwinger equation with a complex "mass". To this end, we define three vectors $\mathbf{e}_\omega$ with $\omega = \pm 1, 0$:

$$
\begin{aligned}
\mathbf{e}_{+1} &= -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \qquad \mathbf{e}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \\
\bar{\mathbf{e}}_{+1} &= -\frac{1}{\sqrt{2}}(1,-i,0), \qquad \bar{\mathbf{e}}_0 = (0,0,1), \qquad \bar{\mathbf{e}}_{-1} = \frac{1}{\sqrt{2}}(1,i,0).
\end{aligned}
\tag{2.12}
$$

Further, define

$$
\begin{aligned}
f^\mu(P,\omega) &= \left(\frac{\mathbf{e}_\omega\cdot\mathbf{P}}{E_R}, \mathbf{e}_\omega + \frac{\mathbf{P}(\mathbf{e}_\omega\cdot\mathbf{P})}{E_R(P^0+E_R)}\right), \\
\bar{f}^\mu(P,\omega) &= \left(\frac{\bar{\mathbf{e}}_\omega\cdot\mathbf{P}}{E_R}, \bar{\mathbf{e}}_\omega + \frac{\mathbf{P}(\bar{\mathbf{e}}_\omega\cdot\mathbf{P})}{E_R(P^0+E_R)}\right).
\end{aligned}
\tag{2.13}
$$

The Rarita–Schwinger wave functions are given by group-theoretical expressions corresponding to the addition of spins 1 and 1/2:

$$
\begin{aligned}
u^\mu(P,\lambda) &= \sum_{\omega,\varepsilon}\langle 1\omega 1/2\varepsilon|3/2\lambda\rangle f^\mu(P,\omega)u(P,\varepsilon), \\
\bar{u}^\mu(P,\lambda) &= \sum_{\omega,\varepsilon}\langle 1\omega 1/2\varepsilon|3/2\lambda\rangle \bar{u}(P,\varepsilon)\bar{f}^\mu(P,\omega),
\end{aligned}
\tag{2.14}
$$

with $\langle\ldots\rangle$ the appropriate Clebsch–Gordan coefficients. These wave functions obey the equations

$$
\begin{aligned}
(\not{P} - E_R)u^\mu(P,\lambda) &= 0, \qquad P_\mu u^\mu(P,\lambda) = \gamma_\mu u^\mu(P,\lambda) = 0, \\
\bar{u}^\mu(P,\lambda)(\not{P} - E_R) &= 0, \qquad \bar{u}^\mu(P,\lambda)P_\mu = \bar{u}^\mu(P,\lambda)\gamma_\mu = 0,
\end{aligned}
\tag{2.15}
$$

and the identities

$$\sum_\lambda u^\mu(P,\lambda)\bar u^\nu(P,\lambda) = -\frac{\not P + E_R}{2E_R}\left(g^{\mu\nu} - \frac{1}{3}\gamma^\mu\gamma^\nu - \frac{2P^\mu P^\nu}{3E_R^2} + \frac{P^\mu\gamma^\nu - P^\nu\gamma^\mu}{3E_R}\right),$$
$$\sum_\mu \bar u^\mu(P,\lambda)u_\mu\big(P,\lambda'\big) = -2E_R\delta_{\lambda\lambda'}. \tag{2.16}$$

Below, we shall restrict ourselves to the case of the $\Delta N\gamma^*$ transition. However, the formalism can be directly generalized to particles with any spin. The three-point function for this transition takes the form

$$\begin{aligned} F^{\mu\rho}(P,Q) &= i^2\int d^4X d^4Y e^{iPX-iQY}\langle 0|T O^\mu(X)J^\rho(0)\bar\psi(Y)|0\rangle \\ &= \frac{Z_R^{1/2}}{s_R-P^2}\frac{Z_N^{1/2}}{m_N^2-Q^2}\sum_{\lambda,\varepsilon}u^\mu(P,\lambda)\langle P,\lambda|J^\rho(0)|Q,\varepsilon\rangle\bar u(Q,\varepsilon)+\cdots. \end{aligned} \tag{2.17}$$

Here, $O^\mu(X)$ and $\psi(Y)$ denote the Δ and nucleon interpolating field operators, respectively, J^ρ is the electromagnetic current, s_R is the Δ-resonance pole position in the complex plane, and m_N is the nucleon mass. The sum runs over the Δ and nucleon spin projections: $\lambda = -3/2, -1/2, 1/2, 3/2$ and $\varepsilon = -1/2, 1/2$. As already stated after Eq. (2.5), the matrix element that appears on the r.h.s. of the above equation is a mere notation: there is no stable Δ-state in the Fock space of the theory. We shall use this notation throughout the paper.

Projecting out the matrix element from Eq. (2.17), we get

$$\begin{aligned} \langle P,\lambda|J^\rho(0)|Q,\varepsilon\rangle = \lim_{P^2\to s_R,\,Q^2\to m_N^2} &\frac{Z_R^{-1/2}}{2E_R}\frac{Z_N^{-1/2}}{2m_N}\big(s_R-P^2\big)\big(m_N^2-Q^2\big) \\ &\times \bar u_\mu(P,\lambda)F^{\mu\rho}(P,Q)u(Q,\varepsilon). \end{aligned} \tag{2.18}$$

This matrix element can be expressed in terms of three scalar form factors (see, e.g., [47,48])

$$\begin{aligned} \langle P,\lambda|J^\rho(0)|Q,\varepsilon\rangle = \left(\frac{2}{3}\right)^{1/2} &\bar u_\mu(P,\lambda)\big\{G_M(t)\mathcal{K}_M^{\mu\rho} + G_E(t)\mathcal{K}_E^{\mu\rho} + G_C(t)\mathcal{K}_C^{\mu\rho}\big\} \\ &\times u(Q,\varepsilon), \end{aligned} \tag{2.19}$$

where[1]

$$\begin{aligned} \mathcal{K}_M^{\mu\rho} &= -\frac{3}{(E_R+m_N)^2-t}\frac{E_R+m_N}{2m_N}\epsilon^{\mu\rho\alpha\nu}p_\alpha q_\nu, \\ \mathcal{K}_E^{\mu\rho} &= -\mathcal{K}_M^{\mu\rho} + 6i\Delta^{-1}(t)\frac{E_R+m_N}{m_N}\gamma_5\epsilon^{\mu\sigma\alpha\beta}p_\alpha q_\beta\epsilon^{\rho\omega\nu\delta}g_{\sigma\omega}p_\nu q_\delta, \\ \mathcal{K}_C^{\mu\rho} &= 3i\Delta^{-1}(t)\frac{E_R+m_N}{m_N}\gamma_5 q^\mu\big(q^2p^\rho - (q\cdot p)q^\rho\big), \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &p = \frac{1}{2}(P+Q), \qquad q = P-Q, \\ &\Delta(t) = \big((E_R+m_N)^2-t\big)\big((E_R-m_N)^2-t\big). \end{aligned} \tag{2.21}$$

[1] For the Dirac matrices, we use the conventions of Ref. [49].

$K_{M,E,C}$ are, respectively, the magnetic dipole, electric quadrupole and Coulomb (longitudinal) quadrupole covariants. In order to determine these three scalar form factors separately, it is convenient to work in a special kinematics. We choose both 3-momenta $\mathbf{P}$ and $\mathbf{Q}$ along the third axis. Further, the formulae simplify considerably in the rest-frame of the resonance $\mathbf{P}=0$. Below, we shall adopt this choice.

Using Eqs. (2.19) and (2.20), it is straightforward to show that, in the rest-frame of the Δ-resonance,

$$\begin{aligned}
\langle 1/2|J^3(0)|1/2\rangle &= i\,\frac{E_R-Q^0}{E_R}AG_C(t),\\
\langle 1/2|J^+(0)|-1/2\rangle &= -i\sqrt{\frac{1}{2}}A\big(G_M(t)-3G_E(t)\big),\\
\langle 3/2|J^+(0)|1/2\rangle &= -i\sqrt{\frac{3}{2}}A\big(G_M(t)+G_E(t)\big),
\end{aligned}\tag{2.22}$$

where

$$J^+(0)=\frac{J^1(0)+iJ^2(0)}{\sqrt{2}},\qquad A=\frac{E_R+m_N}{2m_N}\sqrt{2E_R\big(Q^0-m_N\big)}.\tag{2.23}$$

Further, it can be shown that the following field operators

$$\begin{aligned}
O_{3/2}(X) &= \frac{1}{2}(1+\Sigma_3)\frac{1}{2}(1+\gamma_0)\frac{1}{\sqrt{2}}\big(O^1(X)-i\Sigma_3O^2(X)\big),\\
O_{1/2}(X) &= \frac{1}{2}(1-\Sigma_3)\frac{1}{2}(1+\gamma_0)\frac{1}{\sqrt{2}}\big(O^1(X)+i\Sigma_3O^2(X)\big),\\
\tilde{O}_{1/2}(X) &= \frac{1}{2}(1+\Sigma_3)\frac{1}{2}(1+\gamma_0)O^3(X)
\end{aligned}\tag{2.24}$$

produce the Δ-particles with spin projection $\lambda=3/2$ and $\lambda=1/2$, respectively. Here, Σ_3 denotes the 4×4 matrix, describing the spin projection on the third axis. In terms of the Pauli matrices, it is given by $\Sigma_3=\mathrm{diag}(\sigma_3,\sigma_3)$. Note also that the operators $O_{3/2}$ and $O_{1/2}$, $\tilde{O}_{1/2}$ belong to the different irreducible representations (irreps), G_2 and G_1, respectively, of the little group of the double cover of the cubic group, corresponding to the boost momentum along the third axis [16]. Moreover, there are two different operators that correspond to the spin projection $\lambda=1/2$, whereas there exists only one operator for the projection $\lambda=3/2$, see Eqs. (113) and (114) of Ref. [16].

The field operators, projecting onto the states with a given third component of the nucleon, are constructed trivially:

$$\bar{\psi}_{\pm 1/2}(Y)=\bar{\psi}(Y)\frac{1}{2}(1\pm\Sigma_3)\frac{1}{2}(1+\gamma_0).\tag{2.25}$$

Using the above operators, we may construct the following three-point functions:

$$\begin{aligned}
\tilde{F}_{1/2}(P,Q) &= i^2\int d^4Xd^4Ye^{iPX-iQY}\langle 0|T\tilde{O}_{1/2}(X)J^3(0)\bar{\psi}_{1/2}(Y)|0\rangle,\\
F_{1/2}(P,Q) &= i^2\int d^4Xd^4Ye^{iPX-iQY}\langle 0|TO_{1/2}(X)J^+(0)\bar{\psi}_{-1/2}(Y)|0\rangle,\\
F_{3/2}(P,Q) &= i^2\int d^4Xd^4Ye^{iPX-iQY}\langle 0|TO_{3/2}(X)J^+(0)\bar{\psi}_{1/2}(Y)|0\rangle.
\end{aligned}\tag{2.26}$$

In the vicinity of the double pole, these functions behave as

$$\mathrm{Tr}\big(\tilde{F}_{1/2}(P,Q)\big)=i\left(\frac{2}{3}\right)^{1/2}\frac{Z_R^{1/2}}{s_R-P^2}\frac{Z_N^{1/2}}{m_N^2-Q^2}\frac{E_R-Q^0}{E_R}BG_C(t)+\cdots,$$
$$\mathrm{Tr}\big(F_{1/2}(P,Q)\big)=i\left(\frac{2}{3}\right)^{1/2}\frac{Z_R^{1/2}}{s_R-P^2}\frac{Z_N^{1/2}}{m_N^2-Q^2}\frac{1}{2}B\big(G_M(t)-3G_E(t)\big)+\cdots,$$
$$\mathrm{Tr}\big(F_{3/2}(P,Q)\big)=i\left(\frac{2}{3}\right)^{1/2}\frac{Z_R^{1/2}}{s_R-P^2}\frac{Z_N^{1/2}}{m_N^2-Q^2}\frac{3}{2}B\big(G_M(t)+G_E(t)\big)+\cdots, \tag{2.27}$$

where the trace is performed over the Dirac indices, and

$$B=\frac{E_R(E_R+m_N)}{m_N}|\mathbf{Q}|. \tag{2.28}$$

So, with a special choice of the interpolating operators, the problem of a particle with spin boils down to the spinless case, considered in the beginning of this section. The three form factors G_C, G_M, G_E can be projected out individually.

3. Extracting the form factors on the lattice

Below, we adapt the formulae of the previous section and formulate the rules for projecting out the form factors G_C, G_M, G_E from the Euclidean Green functions on the lattice. Let us first restrict ourselves to the case when the Δ is stable and consider the following three-point functions at $t'>0$, $t<0$:

$$\tilde{R}_{1/2}(t',t)=\langle 0|\tilde{\mathcal{O}}_{1/2}(t')J^3(0)\bar{\psi}^{\mathbf{Q}}_{1/2}(t)|0\rangle,$$
$$R_{1/2}(t',t)=\langle 0|\mathcal{O}_{1/2}(t')J^+(0)\bar{\psi}^{\mathbf{Q}}_{-1/2}(t)|0\rangle,$$
$$R_{3/2}(t',t)=\langle 0|\mathcal{O}_{3/2}(t')J^+(0)\bar{\psi}^{\mathbf{Q}}_{1/2}(t)|0\rangle, \tag{3.1}$$

where

$$\tilde{\mathcal{O}}_{1/2}(t')=\sum_{\mathbf{X}}\tilde{O}_{1/2}(\mathbf{X},t'),$$
$$\mathcal{O}_{1/2}(t')=\sum_{\mathbf{X}}O_{1/2}(\mathbf{X},t'),$$
$$\mathcal{O}_{3/2}(t')=\sum_{\mathbf{X}}O_{3/2}(\mathbf{X},t'),$$
$$\bar{\psi}^{\mathbf{Q}}_{\pm 1/2}(t)=\sum_{\mathbf{X}}e^{i\mathbf{Q}\mathbf{X}}\bar{\psi}_{\pm 1/2}(\mathbf{X},t). \tag{3.2}$$

The operators $\tilde{\mathcal{O}}_{1/2}(t'), \mathcal{O}_{1/2}(t'), \mathcal{O}_{3/2}(t')$ describe the Δ at rest, whereas $\bar{\psi}^{\mathbf{Q}}_{\pm 1/2}(t)$ corresponds to the nucleon moving with the 3-momentum $\mathbf{Q}$. The operators on the r.h.s. of Eq. (3.2) are given in Eq. (2.24) with the substitution $\gamma_0\to\gamma_4$.

In the limit $t' \to +\infty$, $t \to -\infty$ only the one-particle Δ and nucleon states contribute:

$$\tilde{R}_{1/2}\big(t',t\big) \to \frac{e^{-E_\Delta t'+E_N t}}{4E_\Delta E_N}\langle 0|\tilde{\mathcal{O}}_{1/2}(0)|1/2\rangle\langle 1/2|J^3(0)|1/2\rangle\langle 1/2|\bar{\psi}^{\mathbf{Q}}_{1/2}(0)|0\rangle,$$
$$R_{1/2}\big(t',t\big) \to \frac{e^{-E_\Delta t'+E_N t}}{4E_\Delta E_N}\langle 0|\mathcal{O}_{1/2}(0)|1/2\rangle\langle 1/2|J^+(0)|-1/2\rangle\langle -1/2|\bar{\psi}^{\mathbf{Q}}_{-1/2}(0)|0\rangle,$$
$$R_{3/2}\big(t',t\big) \to \frac{e^{-E_\Delta t'+E_N t}}{4E_\Delta E_N}\langle 0|\mathcal{O}_{3/2}(0)|3/2\rangle\langle 3/2|J^+(0)|1/2\rangle\langle 1/2|\bar{\psi}^{\mathbf{Q}}_{1/2}(0)|0\rangle, \tag{3.3}$$

where $E_\Delta = m_\Delta$ in the rest-frame of the Δ and $E_N = \sqrt{m_N^2 + \mathbf{Q}^2}$ (here, m_Δ denotes the mass of a stable Δ). Further, we define the following 2-point functions

$$\tilde{D}_{1/2}(t) = \mathrm{Tr}\langle 0|\tilde{\mathcal{O}}_{1/2}(t)\bar{\tilde{\mathcal{O}}}_{1/2}(0)|0\rangle,$$
$$D_{1/2}(t) = \mathrm{Tr}\langle 0|\mathcal{O}_{1/2}(t)\bar{\mathcal{O}}_{1/2}(0)|0\rangle,$$
$$D_{3/2}(t) = \mathrm{Tr}\langle 0|\mathcal{O}_{3/2}(t)\bar{\mathcal{O}}_{3/2}(0)|0\rangle,$$
$$D^{\pm}_{\mathbf{Q}}(t) = \mathrm{Tr}\langle 0|\psi^{\mathbf{Q}}_{\pm 1/2}(t)\bar{\psi}^{\mathbf{Q}}_{\pm 1/2}(0)|0\rangle. \tag{3.4}$$

It can be straightforwardly seen that, in the limit $t' \to +\infty$, $t \to -\infty$,

$$\mathcal{N}\frac{\mathrm{Tr}(\tilde{R}_{1/2}(t',t))}{\tilde{D}_{1/2}(t'-t)}\left(\frac{D^+_{\mathbf{Q}}(t')\tilde{D}_{1/2}(-t)\tilde{D}_{1/2}(t'-t)}{\tilde{D}_{1/2}(t')D^+_{\mathbf{Q}}(-t)D^+_{\mathbf{Q}}(t'-t)}\right)^{1/2} \to \langle 1/2|J^3(0)|1/2\rangle,$$
$$-\mathcal{N}\frac{\mathrm{Tr}(R_{1/2}(t',t))}{D_{1/2}(t'-t)}\left(\frac{D^-_{\mathbf{Q}}(t')D_{1/2}(-t)D_{1/2}(t'-t)}{D_{1/2}(t')D^-_{\mathbf{Q}}(-t)D^-_{\mathbf{Q}}(t'-t)}\right)^{1/2} \to \langle 1/2|J^+(0)|-1/2\rangle,$$
$$-\mathcal{N}\frac{\mathrm{Tr}(R_{3/2}(t',t))}{D_{3/2}(t'-t)}\left(\frac{D^+_{\mathbf{Q}}(t')D_{3/2}(-t)D_{3/2}(t'-t)}{D_{3/2}(t')D^+_{\mathbf{Q}}(-t)D^+_{\mathbf{Q}}(t'-t)}\right)^{1/2} \to \langle 3/2|J^+(0)|1/2\rangle, \tag{3.5}$$

where $\mathcal{N} = \sqrt{4E_\Delta E_N}$ and the Euclidean analogs of Eqs. (2.22) read (cf., e.g., with Ref. [3][2])

$$\langle 1/2|J^3(0)|1/2\rangle = \frac{E_\Delta - Q^0}{E_\Delta} A G_C(t)$$
$$\langle 1/2|J^+(0)|-1/2\rangle = \sqrt{\frac{1}{2}} A\big(G_M(t) - 3G_E(t)\big),$$
$$\langle 3/2|J^+(0)|1/2\rangle = \sqrt{\frac{3}{2}} A\big(G_M(t) + G_E(t)\big), \tag{3.6}$$

where $t = (E_\Delta - E_N)^2 - \mathbf{Q}^2$ and the quantity A is given by Eq. (2.23) with the replacement $E_R \to E_\Delta$. We would like to also mention that, in case of a stable Δ, the above relations hold up to the Lorentz-non-invariant terms exponentially suppressed in a box of size L.

When the Δ becomes unstable, the interpretation of the above equations changes. The ratios given in Eq. (3.5) can be still formed, but the functions that are extracted from these ratios are the matrix elements of the electromagnetic current calculated between a certain eigenstate of the Hamiltonian in a finite volume (with the volume-dependent energy E_Δ) and a one-nucleon state.

[2] Note the difference in sign and in a factor 2 with the third line of Eq. (4) of Ref. [3].

The normalization constant $\mathcal{N}$ in these equations also changes. Namely, $\mathcal{N} = \sqrt{8w_{1\Delta}w_{2\Delta}E_N}$, where $w_{1\Delta}$ and $w_{2\Delta}$ are the energies of the nucleon and a pion in the CM system: $w_{1\Delta} = (E_\Delta^2 + m_N^2 - M_\pi^2)/(2E_\Delta)$, $w_{2\Delta} = (E_\Delta^2 - m_N^2 + M_\pi^2)/(2E_\Delta)$ and $w_{1\Delta} + w_{2\Delta} = E_\Delta$. In the infinite-volume limit, these matrix elements do not coincide with the resonance matrix elements defined in the previous section. Rather, the energy of any fixed level tends to the threshold value in this limit. Note also that, at a given energy, there may exist several eigenstates of the Hamiltonian in the vicinity of the resonance energy, and it is not clear, which of these matrix elements should be identified with the resonance matrix element we are looking for.

It is evident that the situation closely resembles the determination of the resonance pole position from the lattice data by using the Lüscher equation. As it is well known, in order to achieve the goal, one has to perform an analytic continuation into the complex energy plane. A detailed discussion of this procedure can be found, e.g., in Ref. [40]. Below, we give a short description of the procedure. One first extracts the scattering phase shift at different energies from the measured energy spectrum and fits the quantity $p^3 \cot\delta(p)$ by a polynomial in p^2, assuming the effective range expansion (here, p denotes the relative 3-momentum in the CM system). Then, one finds the poles of the T-matrix in the complex plane by finding the zeros of a polynomial with known coefficients. Below we shall prove the generalization of this procedure to the case of the matrix elements. Namely, the matrix elements given in Eq. (3.5) should be measured at several energies (corresponding to the measurement at several volumes), and the result should be fitted by some polynomial. Further, we shall show that, replacing p^2 by p_R^2 in this polynomial, where p_R denotes the value of the relative 3-momentum at the resonance pole, one obtains the resonance form factors defined in the previous section (up to the corrections that are exponentially suppressed in large volumes). This is the main result of our work. Note also that this prescription is much simpler than the one for the elastic Δ-form factor (see Ref. [40]) since, as we shall see below, no finite fixed points arise in the case of the transition form factor.

As it is clear from the previous discussion, in order to perform the fit, the matrix elements should be measured at several values of the relative 3-momentum p. These matrix elements depend on two kinematic variables: apart from p, there is the nucleon 3-momentum $|\mathbf{Q}|$ (alternatively, the variable t which at the resonance takes the complex value $t = (E_R - Q^0)^2 - \mathbf{Q}^2$). Note also that fixing $|\mathbf{Q}|$ is equivalent to fixing t because the (complex) resonance energy E_R, which is defined in the infinite volume, is also fixed. On the lattice, it is more convenient to fix the real quantity $|\mathbf{Q}|$, and we stick to this choice in the following.

According to the previous discussion, our goal is to find a way to "scan" the resonance region in the variable p, leaving the other variable $|\mathbf{Q}|$ fixed. There is, however, a problem, if one performs this scan by doing measurements at different volumes. Namely, the momentum on the lattice along any axis is quantized, and the smallest nonzero momentum available is equal to $2\pi/L$, where L denotes the box size. Consequently, if L is varied, the quantity $|\mathbf{Q}|$ will change along with p.

We can propose at least two strategies that help to circumvent this problem:

1. The use of asymmetric boxes. Consider the box with the geometry $L \times L \times L'$ and direct $\mathbf{Q}$ along the third axis. The Δ is in the rest frame. Changing L does not affect $\mathbf{Q}$ but affects p.
2. Using (partially) twisted boundary conditions. One may apply the twisting to a single quark in the nucleon, namely, the one that is attached to the photon (see Fig. 1). This gives an additional momentum to the nucleon along the third axis. The magnitude of this change is $|\mathbf{Q}_\theta| = \theta/L$. On the other hand, changing the cubic box size, we also change the magnitude of the nucleon momentum $|\mathbf{Q}|$. One may adjust the value of the twisting angle θ so that the

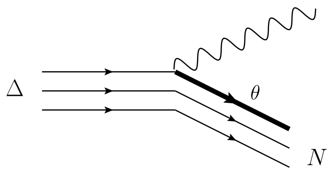

Fig. 1. Twisting a single quark in the nucleon.

sum of these two effects cancels and the nucleon momentum is kept fixed. It is important to stress that the value of θ can be determined prior to the simulations since it depends only on the box size. We also note that a similar technique of twisting has been already applied in the past for the calculation of nucleon form factors [50].

To summarize, on the lattice we have to measure the matrix elements given in Eq. (3.5). These matrix elements depend on two kinematic variables: the relative 3-momentum p in the Δ-channel and the 3-momentum $\mathbf{Q}$ of the nucleon (the three-momentum of the photon is $-\mathbf{Q}$). Using asymmetric boxes or twisted boundary conditions, we may scan the resonance region in p while keeping $\mathbf{Q}$ fixed. Let us now discuss how to perform the analytic continuation into the complex plane and extract the resonance form factors with the use of Eq. (3.6).

4. Matrix elements in a finite volume

4.1. Two-point function

In this section, we shall consider the resonance matrix elements by using the technique of the non-relativistic effective field theory in a finite volume. While doing so, we closely follow the path of Ref. [40], adapting the formulae given there, whenever necessary. To avoid problems, related to the mixing of the partial waves, the Δ-resonance is always considered in the CM frame. In this case, there is no S- and P-wave mixing. Neglecting the (small) P_{31} wave, the Lüscher equation [7] for the P_{33} wave is written as follows:

$$p \cot\delta(p) + p\cot\phi(q) = 0, \qquad q = \frac{pL}{2\pi}, \tag{4.1}$$

where

$$p\cot\phi(q) = -\frac{2}{\sqrt{\pi}L}\left\{\hat{Z}_{00}\big(1;q^2\big) \pm \frac{1}{\sqrt{5}q^2}\hat{Z}_{20}\big(1;q^2\big)\right\}, \tag{4.2}$$

and $\delta(p)$ is the P_{33} phase shift in the infinite volume. Further, $\hat{Z}_{lm}(1;q^2)$ denotes the Lüscher zeta-function (for the asymmetric boxes, in general), and the signs $+$ and $-$ are chosen for the irreps G_1 and G_2 of the little group corresponding to $\mathbf{d} = (0,0,1)$, respectively (see Ref. [16]). In particular, for a symmetric box, $\hat{Z}_{lm}(1;q^2) = Z_{lm}(1;q^2)$ and $Z_{20}(1;q^2) = 0$ in the CM frame (there is no mixing to the P_{31} wave in this case). For an asymmetric box with $L' = xL$,

$$\hat{Z}_{lm}\big(1;q^2\big) = \frac{1}{x}\sum_{\mathbf{n}\in\mathbb{Z}^3}\frac{\mathcal{Y}_{lm}(\mathbf{r})}{\mathbf{r}^2 - q^2}, \qquad r_{1,2} = n_{1,2},\ r_3 = \frac{1}{x}n_3, \qquad \hat{Z}_{20}\big(1;q^2\big) \neq 0. \tag{4.3}$$

The matrix element of an operator $\mathcal{O}_i$ between the vacuum and an eigenstate of a Hamiltonian $\langle 0|\mathcal{O}_i(0)|n\rangle$ contains two-particle reducible diagrams describing initial-state interactions, which

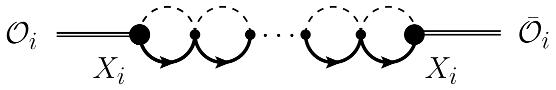

Fig. 2. Initial-state pion–nucleon interactions in the two-point function. The quantity X_i stands for the coupling of the operator $\mathcal{O}_i$ to the pion–nucleon pair in the intermediate state.

are volume-dependent (here, $\mathcal{O}_i$ stands for one of the operators $\tilde{\mathcal{O}}_{1/2}, \mathcal{O}_{1/2}, \mathcal{O}_{3/2}$). This matrix element is proportional to U_i even in case of an unstable Δ, where

$$\begin{aligned}
U_{3/2} &= \frac{1}{2}(1+\Sigma_3)\frac{1}{2}(1+\gamma_4)\frac{1}{\sqrt{2}}\big(u^1(P,3/2) - i\Sigma_3 u^2(P,3/2)\big),\\
U_{1/2} &= \frac{1}{2}(1-\Sigma_3)\frac{1}{2}(1+\gamma_4)\frac{1}{\sqrt{2}}\big(u^1(P,1/2) + i\Sigma_3 u^2(P,1/2)\big),\\
\tilde{U}_{1/2} &= \frac{1}{2}(1+\Sigma_3)\frac{1}{2}(1+\gamma_4)u^3(P,1/2).
\end{aligned} \tag{4.4}$$

This fact can be verified straightforwardly, since both the above matrix element as well as U_i are Dirac spinors with only one nonzero entry.

The calculation of the volume-dependent factor in the matrix element proceeds by using the same technique as in the derivation of Eq. (46) of Ref. [40]. Namely, we calculate the two-point function of the operators $\mathcal{O}_i, \bar{\mathcal{O}}_i$ in the non-relativistic effective field theory below the inelastic threshold. According to the discussion above, the coupling of the operator $\mathcal{O}_i$ to the pion–nucleon state in the effective theory is described by a local vertex $X_i U_i$, where the scalar function $X_i = X_i^{(0)} + X_i^{(1)}\mathbf{p}^2 + \cdots$ contains the terms with $0, 2, \ldots$ derivatives. Here, $\mathbf{p}^2$ stands for the relative momentum squared of the pion–nucleon pair in the CM system. Note that the X_i contain only short-range physics and are the same in a finite and in the infinite volume. Explicit values of $X_i^{(m)}$ are not important, because the X_i cancel in the final expressions.

Summing now up the pion–nucleon bubble diagrams as shown in Fig. 2, it is seen that the Euclidean two-point function takes the form

$$\begin{aligned}
&\langle 0|\mathcal{O}_i(x_0)\bar{\mathcal{O}}_i(y_0)|0\rangle\\
&\quad = U_i X_i \left\{ \int_{-\infty}^{\infty} \frac{dP_0}{2\pi} e^{iP_0(x_0-y_0)} V\big(-iJ(P_0) - J^2(P_0)T(P_0)\big)\right\} X_i \bar{U}_i,
\end{aligned} \tag{4.5}$$

where

$$J(P_0) = \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{2w_1(\mathbf{k})2w_2(\mathbf{k})}\,\frac{1}{P_0 - i(w_1(\mathbf{k}) + w_2(\mathbf{k}))}, \tag{4.6}$$

$w_1(\mathbf{k}) = \sqrt{m_N^2 + \mathbf{k}^2}$, $w_2(\mathbf{k}) = \sqrt{M_\pi^2 + \mathbf{k}^2}$, V is the lattice volume and $T(P_0)$ denotes the pion–nucleon scattering amplitude in a finite volume[3]

$$T(P_0) = \frac{8\pi\sqrt{s}}{p\cot\delta(p) + p\cot\phi(q)}, \qquad s = -P_0^2, \tag{4.7}$$

and p is the relative momentum in the CM frame, corresponding to the total energy $\sqrt{s}$.

[3] Note that here we use a different normalization in the partial-wave expansion of the T-matrix than in Ref. [40].

Next, we perform the integration over the variable P_0, using Cauchy's theorem. It can be shown that only the poles of $T(P_0)$ contribute to this integral. In the vicinity of the n-th pole, this function is given by

$$T(P_0)=\frac{8\pi E_n^2}{w_{1n}w_{2n}}\frac{\sin^2\delta(p_n)}{\delta'(p_n)+\phi'(q_n)}\frac{1}{E_n+iP_0}+\cdots, \tag{4.8}$$

where E_n are the eigenenergies in the box, p_n is the corresponding relative 3-momentum, $q_n=p_nL/(2\pi)$, and $w_{1n}=(E_n^2+m_N^2-M_\pi^2)/(2E_n)$, $w_{2n}=(E_n^2-m_N^2+M_\pi^2)/(2E_n)$. The derivatives are taken with respect to the variable p, so that $\phi'(q)=d\phi(q)/dp=(L/2\pi)d\phi(q)/dq$.

Performing the integration over p_0 and using the Lüscher equation

$$\frac{1}{V}\sum_{\mathbf{k}}\frac{1}{2w_1(\mathbf{k})2w_2(\mathbf{k})}\frac{1}{w_1(\mathbf{k})+w_2(\mathbf{k})-E_n}=\frac{p_n\cot\delta(p_n)}{8\pi E_n}, \tag{4.9}$$

we finally get

$$\begin{aligned}&\langle 0|\mathcal{O}_i(x_0)\bar{\mathcal{O}}_i(y_0)|0\rangle\\&\quad=U_iX_i\left\{V\sum_n e^{-E_n(x_0-y_0)}\frac{\cos^2\delta(p_n)}{\delta'(p_n)+\phi'(q_n)}\frac{p_n^2}{8\pi w_{1n}w_{2n}}\right\}X_i\bar{U}_i.\end{aligned} \tag{4.10}$$

On the other hand,[4]

$$\langle 0|\mathcal{O}_i(x_0)\bar{\mathcal{O}}_i(y_0)|0\rangle=\sum_n\frac{e^{-E_n(x_0-y_0)}}{4w_{1n}w_{2n}}\langle 0|\mathcal{O}_i(0)|n\rangle\langle n|\bar{\mathcal{O}}_i(0)|0\rangle. \tag{4.11}$$

Comparing these two equations, we finally get

$$\big|\langle 0|\mathcal{O}_i(0)|n\rangle\big|=U_iX_iV^{1/2}\left(\frac{\cos^2\delta(p_n)}{|\delta'(p_n)+\phi'(q_n)|}\frac{p_n^2}{2\pi}\right)^{1/2}. \tag{4.12}$$

4.2. Three-point function

Next, let us consider the current matrix elements in a finite volume $F_i=F_i(p,|\mathbf{Q}|)$, $i=1,2,3$, which appear on the r.h.s. of Eq. (3.5). The derivation which is given below is essentially similar to Eqs. (65)–(70) of Ref. [40]. We start from the calculation of the three-point function, summing up the bubble diagrams in the non-relativistic effective theory. There are two types of diagrams, which are shown in Fig. 3 to which one has to add the diagrams obtained by adding any number of pion loops to the initial states interaction (see Fig. 4). The self-energy insertions in the outgoing nucleon line can be safely ignored since the nucleon is a stable particle (see discussion below) and hence such diagrams lead to the exponentially suppressed contributions in a finite volume. As a result, the matrix element is written as a sum of two contributions

$$\begin{aligned}\langle 0|\mathcal{O}_i(x_0)J^a(0)|Q,\varepsilon\rangle=U_iX_i\int_{-\infty}^{\infty}\frac{dP_0}{2\pi}e^{iP_0x_0}\frac{p\cot\delta(p)}{8\pi\sqrt{s}}T(P_0)\\ \times\big(-iJ(P_0)\bar{F}_i^{(A)}\big(p,|\mathbf{Q}|\big)+\hat{F}_i^{(B)}\big(p,|\mathbf{Q}|\big)\big),\end{aligned} \tag{4.13}$$

[4] The normalization of the one-particle states here differs from the one used in Ref. [40]. Here, we use the normalization $\langle p|q\rangle=2p_0V\delta_{pq}$.

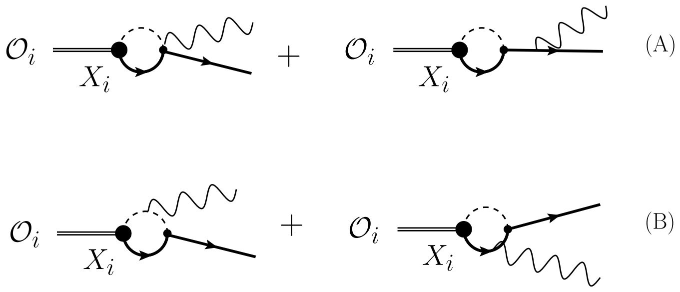

Fig. 3. Typical diagrams contributing to the $\Delta N\gamma^*$ transition form factor: (A) point vertex and emission of the photon from the external nucleon line; (B) emission of the photon from the internal lines. As in Fig. 2, the quantity X_i stands for the coupling of the operator $\mathcal{O}_i$ to the pion–nucleon pair in the intermediate state.

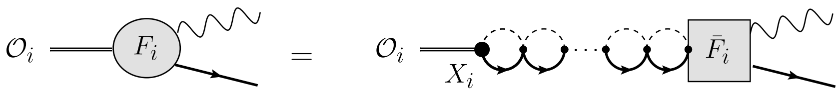

Fig. 4. Initial-state pion–nucleon interactions in the $\Delta N\gamma^*$ transition form factor. The quantity $\bar{F}_i$ denotes the sum of all irreducible diagrams.

where $F_i^{(A)}(p,|\mathbf{Q}|)$ does not depend on L (up to the exponentially suppressed contributions) and the indices a,ε have been suppressed for brevity.

The diagrams of the type (B) can be potentially dangerous. Indeed in the case of the elastic form factor, such diagrams lead to the so-called finite fixed points, see Ref. [40]. However, the fact that one of the external particles (the nucleon) is stable simplifies matters considerably. As shown in Section 4.2.2 (see Eq. (4.39)), in that case the quantity $\hat{F}_i^{(B)}(p,|\mathbf{Q}|)$ can be written as a product of two factors:

$$\hat{F}_i^{(B)}\big(p,|\mathbf{Q}|\big) = -iJ(P_0)\bar{F}_i^{(B)}\big(p,|\mathbf{Q}|\big), \tag{4.14}$$

where $\bar{F}_i^{(B)}(p,|\mathbf{Q}|)$, again, does not depend on L up to the exponentially suppressed contributions. Physically, this corresponds to the Taylor expansion of the pion propagator attached to the nucleon in diagram Fig. 5: this propagator shrinks to a point (a similar discussion holds when the photon is attached to the nucleon line) and the contributions of diagrams (B) to the matrix element are equivalent to the one of diagrams (A).

Consequently, we can rewrite Eq. (4.13)

$$\begin{aligned}&\langle 0|\mathcal{O}_i(x_0)J^a(0)|Q,\varepsilon\rangle\\ &\quad = U_iX_i\int\limits_{-\infty}^{\infty}\frac{dP_0}{2\pi}e^{iP_0x_0}\big(-iJ(P_0)\big)\frac{p\cot\delta(p)}{8\pi\sqrt{s}}T(P_0)\bar{F}_i\big(p,|\mathbf{Q}|\big),\end{aligned} \tag{4.15}$$

where $\bar{F}_i(p,|\mathbf{Q}|) = \bar{F}_i^{(A)}(p,|\mathbf{Q}|) + \bar{F}_i^{(B)}(p,|\mathbf{Q}|)$ denotes the full irreducible amplitude for the $\pi N \to \gamma^* N$ transition.

Performing now the Cauchy integral over P_0 and comparing to the spectral representation of the three-point function, we get

$$\langle 0|\mathcal{O}_i(0)|n\rangle\frac{1}{4w_{1n}w_{2n}}F_i\big(p_n,|\mathbf{Q}|\big) = X_iU_i\frac{\cos^2\delta(p_n)}{\delta'(p_n)+\phi'(q_n)}\frac{p_n^2}{8\pi w_{1n}w_{2n}}\bar{F}_i\big(p_n,|\mathbf{Q}|\big). \tag{4.16}$$

From Eqs. (4.12) and Eq. (4.16) one obtains

$$\left|\bar{F}_i\left(p_n,|\mathbf{Q}|\right)\right| = V^{1/2}\left(\frac{\cos^2\delta(p_n)}{|\delta'(p_n)+\phi'(q_n)|}\frac{p_n^2}{2\pi}\right)^{-1/2}\left|F_i\left(p_n,|\mathbf{Q}|\right)\right|. \tag{4.17}$$

Before we proceed further, an important remark is in order. As already discussed in Section 2, the form factor can be defined either on the real energy axis, or at the resonance pole (although, strictly speaking, only the latter definition is a rigorous one, the former is process-dependent). Below we shall provide the formulae which enable one to "translate" lattice data into one of these definitions.

4.2.1. Real energy axis

On the *real* energy axis, the infinite-volume matrix element, corresponding to the scattering process $\pi N \to \gamma^* N$ at low energies, is given by a geometric series of the pion–nucleon bubbles in the final state. This series sums up into the following expression:

$$\mathcal{A}_i\left(p,|\mathbf{Q}|\right) = \frac{p\cot\delta(p)}{p\cot\delta(p)-ip}\bar{F}_i\left(p,|\mathbf{Q}|\right) = e^{i\delta(p)}\cos\delta(p)\bar{F}_i\left(p,|\mathbf{Q}|\right). \tag{4.18}$$

Here, we have assumed that the quantity $\bar{F}_i(p,|\mathbf{Q}|)$ has a smooth infinite-volume limit (see the proof below). The amplitudes $\mathcal{A}_i$ are proportional to the linear combinations of the so-called transverse magnetic (M), transverse electric (E) and scalar (S) multipoles. The latter appear in the partial wave decomposition of the $\pi N \to \gamma^* N$ scattering amplitude (see Appendix A for the details) and can be extracted from the analysis of the experimental data. The multipoles contain information on the resonance states.[5]

Note that Eq. (4.18) is nothing but Watson's theorem, which indeed holds near the Δ-resonance in the elastic region. At a first glance, $\mathcal{A}_i(p,|\mathbf{Q}|)$ vanishes when $\delta = 90°$. In order to show that this is not the case, we rewrite Eq. (4.18) as follows:

$$\mathcal{A}_i\left(p,|\mathbf{Q}|\right) = \frac{e^{i\delta(p)}}{p^3}\sin\delta(p)p^3\cot\delta(p)\bar{F}_i\left(p,|\mathbf{Q}|\right), \tag{4.19}$$

Assuming that the effective-range expansion holds in the resonance region, one may write

$$p^3\cot\delta(p) \doteq h\left(p^2\right) = -\frac{1}{a}+\frac{1}{2}rp^2+\cdots, \tag{4.20}$$

where a is the P-wave scattering volume and r is the effective range. The function $h(p^2)$ should have a zero at $p^2 = p_A^2$, where the scattering phase passes through 90°. It is easy to get convinced that the quantity $\bar{F}_i(p,|\mathbf{Q}|)$ should have a pole exactly at the same value of p^2. This pole corresponds to the exchange of the bare Δ in the s-channel (since our effective non-relativistic Lagrangian does not include the explicit Δ, the pole will manifest itself in the divergence on the perturbative series at $p^2 = p_A^2$). Consequently, not the quantity $\bar{F}_i(p,|\mathbf{Q}|)$ alone, but the product $p^3\cot\delta(p)\bar{F}_i(p,|\mathbf{Q}|)$ is a low-energy polynomial that can be safely expanded in the resonance region and that, in general, does not vanish at $p^2 = p_A^2$. It follows from this that the multipoles $\mathcal{A}_i$, defined by Eq. (4.19), take finite values at the resonance.

Finally, combining the above equations, we arrive at an analogue of the Lüscher–Lellouch equation for the photoproduction amplitude in the elastic region

[5] We would like to mention here that in Ref. [51] the calculation of the deuteron photodisintegration amplitude in a finite volume was addressed by using a slightly different technique.

$$\mathcal{A}_i\big(p_n,|\mathbf{Q}|\big)=e^{i\delta(p_n)}\,V^{1/2}\biggl(\frac{1}{|\delta'(p_n)+\phi'(q_n)|}\frac{p_n^2}{2\pi}\biggr)^{-1/2}\big|F_i\big(p_n,|\mathbf{Q}|\big)\big|. \tag{4.21}$$

Eq. (4.21) comprises one of the main results of the present article. It allows to extract the multipole amplitudes from lattice data.

In order to obtain the $\Delta N\gamma^*$ matrix elements F_i^A, defined on the real axis, one parameterizes the imaginary parts of the multipoles through the matrix element of the electromagnetic current between N and Δ states (see, e.g., Ref. [43]). Then, in the narrow width approximation, for the amplitudes $\mathcal{A}_i(p,|\mathbf{Q}|)$ we get:

$$\big|\mathrm{Im}\,\mathcal{A}_i\big(p_A,|\mathbf{Q}|\big)\big|=\sqrt{\frac{8\pi}{p_A\Gamma}}\big|F_i^A\big(p_A,|\mathbf{Q}|\big)\big|, \tag{4.22}$$

where all quantities are *real* and taken at the Breit–Wigner pole $p=p_A$, the Γ is the total width of the Δ-resonance.

4.2.2. Complex energy plane

Next, we consider the extraction of the form factor at the resonance pole. This implies the analytic continuation of the above result into the complex p-plane. In order to do this, let us first consider the two-point function in the infinite volume

$$\langle 0|O_i(x)\bar{O}_i(y)|0\rangle=\int\frac{d^4P}{(2\pi)^4}e^{iP(x-y)}D_i\big(P^2\big), \tag{4.23}$$

where, in the CM system $P_\mu=(P_0,\mathbf{0})$ the quantity $D_i(P^2)$ takes the form (cf. Eq. (4.5))

$$D_i\big(P^2\big)=U_iX_i\big(-iJ_\infty(P_0)-J_\infty^2(P_0)T_\infty(P_0)\big)X_i\bar{U}_i. \tag{4.24}$$

Here, the quantities $J_\infty(P_0)$ and $T_\infty(P_0)$ denote the infinite-volume counterparts of the quantities defined by Eqs. (4.6) and (4.7). In the Minkowski space, with $P_0=i\sqrt{s}$, these quantities are given by

$$J_\infty(P_0)=-\frac{p}{8\pi\sqrt{s}},\qquad T_\infty(P_0)=\frac{8\pi\sqrt{s}}{p\cot\delta(p)-ip}. \tag{4.25}$$

These expressions are valid on the first Riemann sheet. On the second sheet, the relative momentum p changes sign.

Suppose now that the scattering amplitude $T_\infty(P_0)$ has a pole at $s=s_R$ on the second Riemann sheet. Writing down the effective-range expansion in a form of Eq. (4.20), one first finds the pole position in the complex plane from the equation

$$-\frac{1}{a}+\frac{1}{2}rp_R^2+\cdots=-ip_R^3. \tag{4.26}$$

Further, in the vicinity of the pole $p=p_R$ we get

$$p^2\big(p\cot\delta(p)+ip\big)=(s-s_R)\big(2p_Rh'\big(p_R^2\big)+3ip_R^2\big)\frac{w_{1R}w_{2R}}{2p_Rs_R}+O\big((s-s_R)^2\big) \tag{4.27}$$

where $w_{1R}=\sqrt{m_N^2+p_R^2}$, $w_{2R}=\sqrt{M_\pi^2+p_R^2}$, $s_R=E_R^2=(w_{1R}+w_{2R})^2$ and the derivative of h is taken with respect to the variable p^2. Using this expansion, one may obtain the value of the wave function renormalization constant at the pole:

$$D_i(s) \to U_i X_i \frac{Z_R}{s_R - s} X_i \bar{U}_i + \text{regular terms at } s \to s_R,$$
$$Z_R = \left(\frac{p_R}{8\pi E_R}\right)^2 \left(\frac{16\pi p_R^3 E_R^3}{w_{1R} w_{2R}(2p_R h'(p_R^2) + 3ip_R^2)}\right). \tag{4.28}$$

The three-point function in the infinite volume is given by

$$\langle 0|O_i(x)J^a(0)|Q,\varepsilon\rangle$$
$$= U_i X_i \int \frac{d^4P}{(2\pi)^4} e^{iPx} \big(-iJ_\infty(P_0)\big) \frac{p\cot\delta(p)}{8\pi\sqrt{s}} T_\infty(P_0) \bar{F}_i\big(p, |\mathbf{Q}|\big), \tag{4.29}$$

where $\bar{F}_i(p, |\mathbf{Q}|)$ is the same as in Eq. (4.15).

Next, we assume that the two-particle irreducible part $\bar{F}_i(p_n, |\mathbf{Q}|)$ can be analytically continued into the complex plane. Separating the pole contribution in the three-point function, we get our final expression for the resonance matrix element F_i^R, evaluated at the pole

$$F_i^R\big(p_R, |\mathbf{Q}|\big) = Z_R^{1/2} \bar{F}_i\big(p_R, |\mathbf{Q}|\big). \tag{4.30}$$

As a test of our final formula, let us consider the case when the resonance is infinitely narrow. Then, the pole tends to the real axis, and $E_R \to E_n$, $p_R \to p_n$. Still, we assume that the (real) energy E_R is above the two-particle threshold.

First, we can express the amplitudes $\mathcal{A}_i$ through F_i^R:

$$\mathcal{A}_i\big(p_R, |\mathbf{Q}|\big) = Z_R^{-1/2} F_i^R\big(p_R, |\mathbf{Q}|\big). \tag{4.31}$$

Further, since $h(p^2) = p^3 \cot\delta(p)$, at the resonance we have

$$2p_R h'\big(p_R^2\big) + 3ip_R^2 = -\frac{p_R^3 \delta'(p_R)}{\sin^2\delta(p_R)}. \tag{4.32}$$

Moreover, in the vicinity of an infinitely narrow resonance, the derivative of the phase shift behaves as

$$\delta'(p_R) = \frac{2}{\Gamma} \frac{p_R E_R}{w_{1R} w_{2R}}, \tag{4.33}$$

where Γ is the Breit–Wigner width of the resonance. The renormalization constant Z_R becomes

$$Z_R = -\frac{p_R \Gamma}{8\pi}, \tag{4.34}$$

and we finally obtain

$$\big|\mathrm{Im}\,\mathcal{A}_i\big(p_R, |\mathbf{Q}|\big)\big| = \sqrt{\frac{8\pi}{p_R \Gamma}} \big|F_i^R\big(p_R, |\mathbf{Q}|\big)\big|, \tag{4.35}$$

where $p_R \to p_A$. This formula coincides exactly with Eq. (4.22).

Further, in the limit considered, the derivative of the phase shift explodes (see Eq. (4.33)) whereas the quantity $\phi'(q_n)$ stays finite. Consequently, one may neglect $\phi'(q_n)$ in all formulae. Expressing the quantity $\bar{F}_i(p_R, |\mathbf{Q}|)$ through $F_i(p_R, |\mathbf{Q}|)$ by using Eq. (4.17) and substituting in Eq. (4.30), we arrive at a fairly simple result in this limit:

$$F_i^R\big(p_n, |\mathbf{Q}|\big) = V^{1/2} \left(\frac{E_n}{2w_{1n} w_{2n}}\right)^{1/2} F_i\big(p_n, |\mathbf{Q}|\big). \tag{4.36}$$

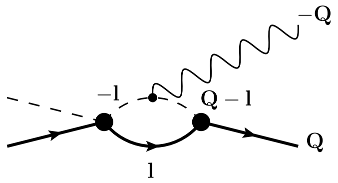

Fig. 5. A potentially dangerous diagram which involves the irreducible vertex $\bar{F}_i$. A diagram where the photon is attached to the nucleon line, can be treated similarly. The case when the photon is attached to a local $\pi N \to \gamma^* N$ vertex is trivial, because, obviously, the corresponding diagram is a low-energy polynomial.

Note that the factor in front of F_i on the r.h.s. of the above equation exactly accounts for the difference in the normalization of the one- and two-particle states in a finite volume (we remind the reader that the energy E_R lies above threshold, so that the resonance still decays into the two-particle state, albeit with an infinitesimally small rate).

4.3. Analytic continuation of the loop diagram

Finally, we want to demonstrate that the analytic continuation of the quantity $\bar{F}_i(p_n, |\mathbf{Q}|)$ into the complex plane is possible. In case of the elastic form factor, this procedure has led to serious difficulties due the existence of the so-called finite fixed points [40]. However, such difficulties do not arise in case of the transition form factors, as can be easily seen by considering the potentially dangerous triangle diagram where the photon is attached to the pion line (see Fig. 5). For simplicity, we neglect all numerators, which are low-energy polynomials. In the rest-frame of the Δ-resonance, the diagram shown in Fig. 5 is equal to

$$I = \frac{1}{V}\sum_{\mathbf{l}} \frac{1}{8w_1(\mathbf{l})w_2(-\mathbf{l})w_2(\mathbf{Q}-\mathbf{l})}\,\frac{1}{(w_1(\mathbf{l})+w_2(-\mathbf{l})-E_n)(w_1(\mathbf{l})+w_2(\mathbf{Q}-\mathbf{l})-Q^0)}. \tag{4.37}$$

This diagram can be simplified by using the following algebraic identity (see Ref. [39])

$$\frac{1}{4w_1(\mathbf{l})w_2(-\mathbf{l})}\,\frac{1}{(w_1(\mathbf{l})+w_2(-\mathbf{l})-E_n)} = \frac{1}{2E_n(\mathbf{l}^2-p_n^2)} + \text{non-singular terms.} \tag{4.38}$$

Since the second denominator in Eq. (4.37) is non-singular, up to the exponentially suppressed terms we get

$$I = \frac{1}{V}\sum_{\mathbf{l}} \frac{1}{2E_n(\mathbf{l}^2-p_n^2)}\,\frac{1}{2}\int_{-1}^{1} dy\,\frac{1}{2\hat{w}_2(\hat{w}_1+\hat{w}_2-Q^0)}, \tag{4.39}$$

where

$$\hat{w}_1 = \sqrt{m_N^2+p_n^2}, \qquad \hat{w}_2 = \sqrt{M_\pi^2+p_n^2+\mathbf{Q}^2-2|\mathbf{Q}|p_n y}. \tag{4.40}$$

Since the first factor on the r.h.s. of Eq. (4.39) can be replaced by $p_n \cot\delta(p_n)$ from the Lüscher equation, and the remaining integral over y is a low-energy polynomial, we see that the quantity $p^2 I$ is a low-energy polynomial in p^2 as well. No finite fixed points arise and the analytic continuation can be performed without any problem. Moreover, there exist only exponentially suppressed corrections to the infinite-volume limit. Finally, it is now easy to check that the quantity $\hat{F}_i^{(B)}(p, |\mathbf{Q}|)$ in Eq. (4.13) can be decomposed as in Eq. (4.14), with the irreducible vertex $\bar{F}_i^{(B)}(p, |\mathbf{Q}|)$ containing only exponentially suppressed finite-volume corrections. The same method can be used for the calculation of the three-point function in the infinite volume, see Eq. (4.29). The sum over the momentum $\mathbf{l}$ in Eq. (4.39) is replaced by the integral and gives a pion–nucleon loop, whereas the remainder is again identified with the irreducible vertex $\bar{F}_i^{(B)}(p, |\mathbf{Q}|)$.

5. A prescription for the measurement of the transition form factors

This section contains a short summary of all our findings. We give a prescription for calculating the $\Delta N\gamma^*$ transition form factors on the lattice, in the rest-frame of the Δ-resonance:

1. The matrix elements $F_i = F_i(p, |\mathbf{Q}|)$ in the right-hand side of Eq. (3.5) are functions of the kinematic variables p and $|\mathbf{Q}|$. Measure these matrix elements at different values of the variable p in the resonance region, keeping the other variable fixed, as explained in Section 3. The scattering phase should be measured at the same values of p.
2. The multipoles for the pion photoproduction are given by

$$\mathcal{A}_i\big(p_n, |\mathbf{Q}|\big) = e^{i\delta(p_n)}\, V^{1/2}\left(\frac{p_n^2}{2\pi\,|\delta'(p_n) + \phi'(q_n)|}\right)^{-1/2}\big|F_i\big(p_n, |\mathbf{Q}|\big)\big|. \tag{5.1}$$

3. The resonance matrix elements, defined at real energies, are proportional to the imaginary part of the multipoles at $p = p_A$, where the phase shift passes through 90°. In the narrow width approximation one has

$$\big|\mathrm{Im}\,\mathcal{A}_i\big(p_A, |\mathbf{Q}|\big)\big| = \sqrt{\frac{8\pi}{p_A \Gamma}}\,\big|F_i^A\big(p_A, |\mathbf{Q}|\big)\big|. \tag{5.2}$$

4. In order to extract the matrix element at the resonance pole, we first multiply each F_i by the pertinent Lüscher–Lellouch factor

$$\bar{F}_i\big(p_n, |\mathbf{Q}|\big) = V^{1/2}\left(\frac{\cos^2\delta(p_n)}{|\delta'(p_n) + \phi'(q_n)|}\,\frac{p_n^2}{2\pi}\right)^{-1/2} F_i\big(p_n, |\mathbf{Q}|\big). \tag{5.3}$$

5. Further, we fit the functions $p^3 \cot\delta(p)\bar{F}_i(p, |\mathbf{Q}|)$ by the effective-range formula

$$p^3 \cot\delta(p)\bar{F}_i\big(p, |\mathbf{Q}|\big) = A_i\big(|\mathbf{Q}|\big) + p^2 B_i\big(|\mathbf{Q}|\big) + \cdots. \tag{5.4}$$

6. Finally, we evaluate the resonance matrix elements by substitution

$$F_i^R\big(p_R, |\mathbf{Q}|\big) = i p_R^{-3} Z_R^{1/2}\big(A_i\big(|\mathbf{Q}|\big) + p_R^2 B_i\big(|\mathbf{Q}|\big) + \cdots\big). \tag{5.5}$$

   The quantities p_R and Z_R should be evaluated separately from the measured phase shifts. Note that the form factors are related to the resonance matrix elements via the formulae given in Eq. (3.6). The kinematic factors in front of the form factors are low-energy polynomials. We emphasize again that, from the two definitions of the resonance matrix elements, given above, only the one which implies the analytic continuation to the resonance pole, yields the result which is process-independent.

6. Conclusions

(i) In this paper, we have formulated an explicit prescription for the measurement of the $\Delta N\gamma^*$ transition form factors on the lattice. The Δ is considered as a resonance, not as a stable particle. The spins of all particles are included, and three different scalar form factors are projected out.

(ii) The framework is based on the use of the non-relativistic effective field theory in a finite volume. This is in accordance with the assumption of validity of the effective range expansion in the vicinity of a resonance that is used for performing the analytic continuation into the complex plane. If this assumption proves to be very restrictive, our approach can be easily adapted for the use of the alternative techniques (e.g., expanding the amplitude in the vicinity of some point near the resonance energy, rather than expanding around threshold).

(iii) The extraction of the elastic resonance form factors from data is a rather subtle procedure due to the presence of the so-called finite fixed points. In case of the transition form factors, considered in the present paper, the method is straightforward. The complexity of the extraction is similar to the one of determining the energy and width of the Δ-resonance. For this reason, we believe that the lattice study of the transition form factors may become feasible in a foreseeable future.

(iv) The extraction of the form factors have been carried out in the rest-frame of the Δ-resonance. There are no serious obstacles to carrying out the same procedure in the moving frames as well, except the mixing between S- and P-waves which takes place, if the Δ is not at the rest.

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Appendix A. Photoproduction amplitudes

In this appendix, we shall establish the connection between our photoproduction amplitude, defined in non-relativistic effective field theory with the relativistic amplitude. Note that in the present paper we deal with the P-wave in the $\gamma^* p \to \pi^0 p$ channel with the total isospin $I = 3/2$ and total spin $J = 3/2$. Below, we give the expressions, using the relativistic normalization of the Dirac spinors [49].

The relativistic photoproduction amplitude can be written in the rest frame of the pion–nucleon system as

$$T = 8\pi E \chi^\dagger(2)\mathcal{F}\chi(1), \tag{A.1}$$

where E is the total energy of the πN system. The Pauli spinors $\chi(1)$, $\chi(2)$ carry the information on the spin states of the nucleons.

The matrix $\mathcal{F}$ has a decomposition (see, e.g., [44])

$$\mathcal{F} = i\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\epsilon} F_1 + (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{q}})\big(\boldsymbol{\epsilon} \cdot (\boldsymbol{\sigma} \times \hat{\boldsymbol{k}})\big) F_2 + i(\tilde{\boldsymbol{q}} \cdot \boldsymbol{\epsilon})(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{k}}) F_3 + i(\tilde{\boldsymbol{q}} \cdot \boldsymbol{\epsilon})(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{q}}) F_4 \\ + i(\hat{\boldsymbol{k}} \cdot \boldsymbol{\epsilon})(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{k}}) F_5 + i(\hat{\boldsymbol{k}} \cdot \boldsymbol{\epsilon})(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{q}}) F_6 - \epsilon_0\big[i(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{q}}) F_7 + i(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{k}}) F_8\big], \tag{A.2}$$

where $\epsilon^\mu = (\epsilon_0, \boldsymbol{\epsilon})$ is the photon polarization vector, $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ and $\hat{\boldsymbol{q}} = \boldsymbol{q}/|\boldsymbol{q}|$ are the unit vectors for the photon and pion momenta respectively, and $\tilde{\boldsymbol{a}} = \boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{k}})\hat{\boldsymbol{k}}$ is a vector with purely transverse components. The eight amplitudes $F_1, \ldots, F_8$ are functions of three independent variables, e.g., the total energy E, the pion angle θ, and the four-momentum squared of the virtual photon, $Q^2 = \boldsymbol{k}^2 - \omega^2 > 0$.

Current conservation additionally implies that

$$|\boldsymbol{k}| F_5 = \omega F_8, \qquad |\boldsymbol{k}| F_6 = \omega F_7. \tag{A.3}$$

Further, the six independent amplitudes $F_1, \ldots, F_6$ have a multipole decomposition

$$\begin{aligned}
F_1 &= \sum_{l \ge 0} \big\{(l M_{l+} + E_{l+}) P'_{l+1} + \big[(l+1) M_{l-} + E_{l-}\big] P'_{l-1}\big\}, \\
F_2 &= \sum_{l \ge 1} \big[(l+1) M_{l+} + l M_{l-}\big] P'_l, \\
F_3 &= \sum_{l \ge 1} \big[(E_{l+} - M_{l+}) P''_{l+1} + ((E_{l-} + M_{l-}) P''_{l-1}\big], \\
F_4 &= \sum_{l \ge 2} (M_{l+} - E_{l+} - M_{l-} - E_{l-}) P''_l, \\
F_5 &= \sum_{l \ge 0} \big[(l+1) L_{1+} P'_{l+1} - l L_{l-} P'_{l-1}\big], \\
F_6 &= \sum_{l \ge 1} \big[l L_{1-} - (l+1) L_{l+}\big] P'_l,
\end{aligned} \tag{A.4}$$

where l is a pion angular momentum and the sign $\pm$ refers to the total spin $J = l \pm 1/2$. The P'_l are the derivatives of the Legendre polynomials $P_l = P_l(\cos\theta)$. Note that in the literature the longitudinal transitions are often described by $S_{l\pm}$ multipoles, related to the $L_{l\pm}$ by

$$S_{l\pm} = |\boldsymbol{k}| L_{l\pm}/\omega. \tag{A.5}$$

In case of scattering in the channel with the quantum numbers of the Δ-resonance, we retain only $\ell\pm = 1+$ partial wave and choose the momentum $\hat{\boldsymbol{k}}$ along the third axis. The pertinent scattering amplitudes then take the form

$$\begin{aligned}
\tilde{T}_{1/2} &= \sqrt{4\pi}\left[\sqrt{\frac{1}{3}} \chi^\dagger_{-1/2} Y_{11}(\hat{\boldsymbol{q}}) + \sqrt{\frac{2}{3}} \chi^\dagger_{1/2} Y_{10}(\hat{\boldsymbol{q}})\right] \chi_{1/2} \tilde{\mathcal{A}}_{1/2}, \\
T_{1/2} &= \sqrt{4\pi}\left[\sqrt{\frac{1}{3}} \chi^\dagger_{-1/2} Y_{11}(\hat{\boldsymbol{q}}) + \sqrt{\frac{2}{3}} \chi^\dagger_{1/2} Y_{10}(\hat{\boldsymbol{q}})\right] \chi_{-1/2} \mathcal{A}_{1/2}, \\
T_{3/2} &= \sqrt{4\pi}\big[\chi^\dagger_{1/2} Y_{11}(\hat{\boldsymbol{q}})\big] \chi_{1/2} \mathcal{A}_{3/2}.
\end{aligned} \tag{A.6}$$

Further, in order to relate the amplitudes $\mathcal{A}_i$ to the multipoles M, E, S, we make the following choice of the polarization vectors

$$
\begin{aligned}
i=1: &\quad \epsilon_0=\frac{|\boldsymbol{k}|}{Q}, \quad \boldsymbol{\epsilon}=\frac{1}{Q}(0,0,\omega), \\
i=2: &\quad \epsilon_0=0, \quad \boldsymbol{\epsilon}=\frac{1}{\sqrt{2}}(1,i,0), \\
i=3: &\quad \epsilon_0=0, \quad \boldsymbol{\epsilon}=\frac{1}{\sqrt{2}}(1,i,0).
\end{aligned}
\tag{A.7}
$$

Below, we shall demonstrate the procedure explicitly for the choice $i=1$. Retaining only the $\ell\pm=1+$ partial wave in Eq. (A.4) and substituting the explicit polarization vectors from Eq. (A.7) into Eq. (A.2), one obtains:

$$
\mathcal{F}=\frac{Q}{|\boldsymbol{k}|}\big[-6i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}})\cos\theta+2i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{q}})\big]S_{1+} \tag{A.8}
$$

The expression $\chi^\dagger(2)\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}})\cos\theta\,\chi(1)$ can be brought into the form

$$
\begin{aligned}
\chi^\dagger(2)(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}})\cos\theta\,\chi(1)=&\ \frac{\sqrt{2}}{3}\sqrt{4\pi}\left[\sqrt{\frac{1}{3}}\chi^\dagger_{-1/2}Y_{11}(\hat{\boldsymbol{q}})+\sqrt{\frac{2}{3}}\chi^\dagger_{1/2}Y_{10}(\hat{\boldsymbol{q}})\right]\chi(1) \\
&-\frac{1}{3}\sqrt{4\pi}\left[\sqrt{\frac{2}{3}}\chi^\dagger_{-1/2}Y_{11}(\hat{\boldsymbol{q}})-\sqrt{\frac{1}{3}}\chi^\dagger_{1/2}Y_{10}(\hat{\boldsymbol{q}})\right]\chi(1).
\end{aligned}
\tag{A.9}
$$

On the other hand, since

$$
\chi^\dagger(2)(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{q}})\chi(1)=\sqrt{4\pi}\left[\sqrt{\frac{2}{3}}\chi^\dagger_{1/2}Y_{10}(\hat{\boldsymbol{q}})-\sqrt{\frac{1}{3}}\chi^\dagger_{-1/2}Y_{11}(\hat{\boldsymbol{q}})\right]\chi(1), \tag{A.10}
$$

the quantity $\chi^\dagger(2)\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{q}})\chi(1)$ gives a $J=1/2$ contribution only. Consequently, the $J=3/2$ partial wave contribution to the relativistic amplitude of Eq. (A.1) is

$$
\tilde{T}_{1/2}=\left(-16\pi iE\sqrt{2}\frac{Q}{|\boldsymbol{k}|}S_{1+}\right)\sqrt{4\pi}\left[\sqrt{\frac{1}{3}}\chi^\dagger_{-1/2}Y_{11}(\hat{\boldsymbol{q}})+\sqrt{\frac{2}{3}}\chi^\dagger_{1/2}Y_{10}(\hat{\boldsymbol{q}})\right]\chi_{1/2}. \tag{A.11}
$$

Comparing this formula with Eq. (A.6), we finally obtain

$$
\tilde{\mathcal{A}}_{1/2}=-16\pi iE\sqrt{2}\frac{Q}{|\boldsymbol{k}|}S_{1+}. \tag{A.12}
$$

The two other cases in Eq. (A.7) can be considered along the same lines. We get

$$
\begin{aligned}
\mathcal{A}_{1/2}&=-\frac{1}{2}(3E_{1+}+M_{1+})(-16\pi iE), \\
\mathcal{A}_{3/2}&=\frac{\sqrt{3}}{2}(E_{1+}-M_{1+})(-16\pi iE).
\end{aligned}
\tag{A.13}
$$

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